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Universal scaling of the valence quark mass dependence of the chiral condensate

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Abstract

Recently, the Columbia group obtained the valence quark mass dependence of the chiral condensate for lattice QCD simulations with dynamical fermions for a series of couplings close to the critical temperature. In this letter we show that the data for temperatures below T_c and sufficiently small valence quark masses can be rescaled to fall on a universal curve. Using chiral random matrix theory we obtain an exact analytical expression for the shape of this curve. We also discuss the universal scaling behavior of the mass dependence of the mass derivatives of the chiral condensate (scalar susceptibilities).

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Numerical simulations of QCD have contributed greatly to our present understanding of the QCD chiral phase transition (see [1] for a recent review). It is now generally believed that the critical temperature is about 150 MeV, and that, for two light flavors, the phase transition is of second order [2]. In the neighborhood of the critical temperature the particle spectrum changes drastically [3] which may lead to important experimental implications. The hadron masses are closely related to the chiral condensate, the order parameter of the chiral phase transition, which, according to the Banks-Casher formula [5], is proportional to the average spectral density of the Dirac operator in the vicinity of zero.

Unfortunately, the volumes of present day lattice calculations are relatively small. On the other hand, the smallest eigenvalues of the lattice QCD Dirac operator show the most significant volume dependence. In order to extract the chiral condensate reliably from lattice QCD simulations detailed knowledge of the volume dependence of the spectral density near zero is essential. Recent lattice calculations of the Columbia group [4] have probed this part of the spectrum via the so called valence quark mass dependence of the chiral condensate. In this paper we attempt to understand this dependence for masses well below the typical hadronic scale Λ. We also give results for the derivative of the chiral condensate (scalar susceptibilities) in this mass region.

The valence quark mass dependence of the chiral condensate is defined by

$$\Sigma^V(m) = \frac{1}{N}\int_0^\infty \frac{2m\rho(\lambda)d\lambda}{\lambda^2+m^2}, \tag{1}$$

where $\rho(\lambda)$ is the average eigenvalue density of the Dirac operator

$$\rho(\lambda) = \sum_n \langle \delta(\lambda-\lambda_n)\rangle, \tag{2}$$

and N is the total number of eigenvalues of the lattice QCD Dirac operator. The average over gauge field configurations, denoted by $\langle\cdots\rangle$, is weighted by the gluonic action and the fermion determinant with sea quark mass equal to $m_{\rm sea}$. It is clear that, for a finite number of eigenvalues, $\Sigma^V(m) \sim 1/m$ for asymptotically large valence quark masses. For valence quark masses well below the average position of the smallest eigenvalue, $\lambda_{\rm min}$, we expect that $\Sigma^V(m) \sim m$. For masses larger than $\lambda_{\rm min}$ but still sufficiently small we hope to see a plateau which can be identified as the chiral condensate (this assumes a smooth

sea quark mass dependence of the chiral condensate). The usual relation between the chiral condensate Σ and the average spectral density is given by the Banks-Casher [5] formula

$$\Sigma = \frac{\pi \rho(0)}{N}. \tag{3}$$

Here, $\rho(0)$ is defined as the extrapolation to zero of the eigenvalue density many spacings away from zero. This can be done unambiguously if there are many eigenvalues below a typical hadronic scale Λ (a more rigorous definition is obtained by taking the limits $m \to 0$ and $N \to \infty$ in this order). Above this scale the eigenvalue density is dominated by the perturbative eigenvalue density $\sim N\lambda^3$. In lattice calculations the latter dependence has to be adjusted for finite size effects. Because of the existence of a finite chiral condensate the Banks-Casher formula tells us that the smallest eigenvalues of the Dirac operator are spaced as $\sim \pi/N\Sigma$. This allows us to define a different limiting procedure of the eigenvalue density of the Dirac operator, namely, the microscopic spectral density defined by [6]

$$\rho_S(z) = \lim_{N\to\infty} \frac{1}{N\Sigma} \rho(\frac{z}{N\Sigma}). \tag{4}$$

In terms of this density, eq. (1) can be rewritten as

$$\Sigma^V(m) = \int_0^\infty \frac{2mN\Sigma \rho_S(z) dz}{z^2 + m^2N^2\Sigma^2}, \tag{5}$$

which suggests to plot the lattice data as $\Sigma(m)/\Sigma$ versus $mN\Sigma$.

In Fig. 1 the points show the original Columbia data [4] obtained from a $16^3 \times 4$ lattice with dynamical (staggered) fermions ($N_f = 2$) with mass $m_{\rm sea}a = 0.01$ (a is the lattice spacing). Results are given for three different values of β (see label of the figure) below T_c. The smallest eigenvalue is of order $2.5\,10^{-4}$ for $\beta = 5.245$ and increases gradually for larger values of β. A clear plateau is observed for two smallest values of β. It is located around $ma \approx 10^{-3}$. The dotted curves are fits to the lattice data for masses larger than $ma \approx 0.001$. However, we did not fit directly the valence quark mass dependence but used a reasonable fitting function for the eigenvalue density, which allowed us to extract $\rho(0)/N$.

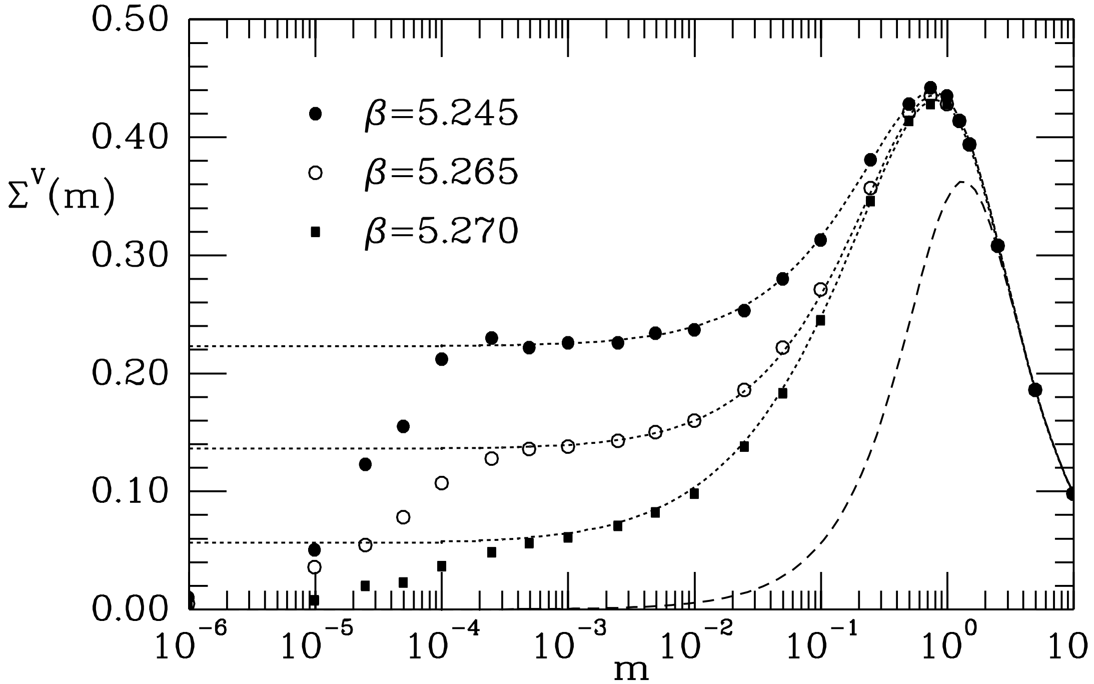

Figure 1: The valence quark mass dependence of the chiral condensate. The points represent the Columbia data for values of β as indicated in the label of the figure. Fits to their data are shown by the dotted curves. The result for the noninteracting Dirac operator is given by the dashed curve.

In Fig. 2 we show the rescaled data for the same values of β. Remarkably, they fall on a single curve suggesting that the microscopic spectral density is a universal function. Below we obtain an analytical expression for the shape of this curve (shown by the full line for $N_f = 0$ in this figure).

From the analysis of spectra of complex systems we know that the correlations between eigenvalues of the Hamiltonian on a scale of no more than a finite number of level spacings can be described by a random matrix theory with the appropriate anti-unitary (time reversal) symmetry [7, 8, 9]. The situation in the case of the Dirac operator is more complicated. First of all, because of the $U_A(1)$-symmetry, in a chiral basis the massless Dirac operator has the block structure (for zero quark masses)

$$\begin{pmatrix} 0 & T \\ T^\dagger & 0 \end{pmatrix}. \tag{6}$$

Since both the gauge fields and the Dirac matrices are in general complex, the anti-unitary

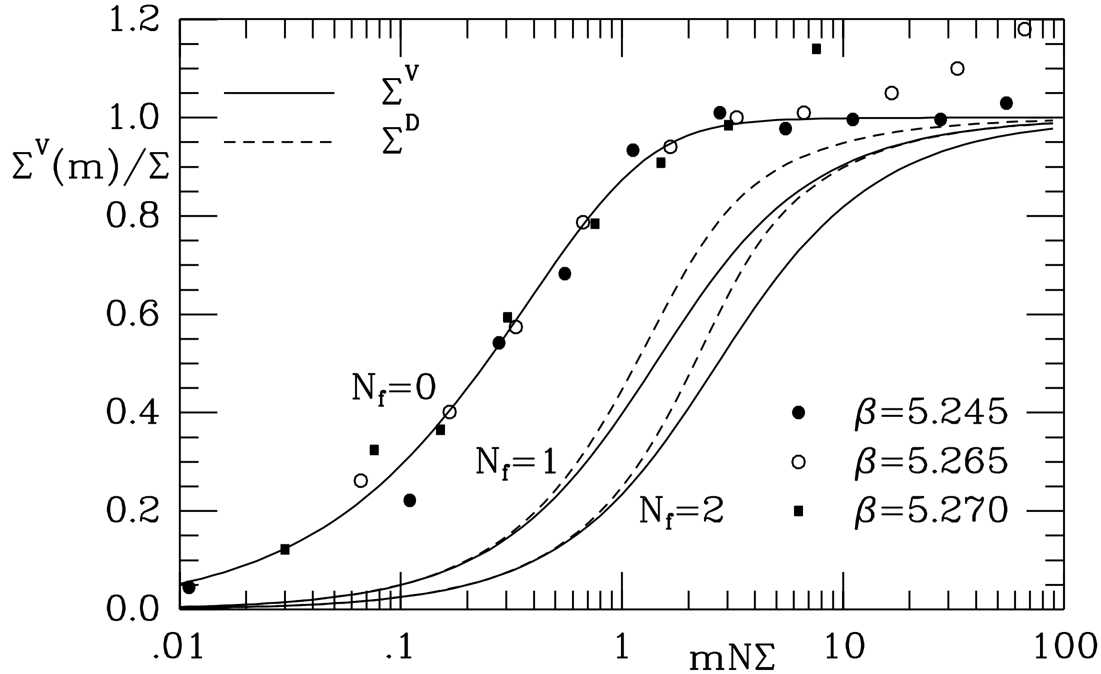

Figure 2: The valence quark mass dependence of the chiral condensate $\Sigma^V(m)$ plotted as $\Sigma^V(m)/\Sigma$ versus $mN\Sigma$. The dots and squares represent lattice results obtained by the Columbia group [1] for values of β as indicated in the label of the figure. The full lines correspond to the random matrix result (eq. (8)) for $\nu = 0$ and N_f as indicated in the label of the figure. Also shown are results (dashed curves) for the quark mass dependence with equal valence and sea quark masses.

symmetries act both in color and Dirac space. For a detailed discussion of this classification we refer to [10]. At this moment we only want to remark that for three colors with fundamental fermions there are no anti-unitary symmetries, leading to complex matrix elements T_{ij}. This ensemble was called the chiral gaussian unitary ensemble (chGUE).

Because of the chiral structure of the Dirac operator, all nonzero eigenvalues occur in pairs $\pm\lambda_n$. Therefore, $\lambda = 0$ is a special point in the spectrum. This allows us consider two types of level correlations: i) microscopic correlations in the bulk of the spectrum; they are given by the Dyson-Mehta-Wigner random matrix ensembles and ii) the microscopic spectral density (defined in (4)). At this moment it is not clear in how far deformations of the random matrix ensemble while keeping its chiral structure (6) lead to the destruction of both types of universal correlations at the same time, or that one of them is more

strongly universal than the other. Up to know we have not been able to construct a random matrix ensemble for which the latter is true [13]. This is an important point because recently Kalkreuter [11] obtained all eigenvalues of the Dirac operator for a 12^4 lattice with dynamical (staggered) fermions. We have analyzed the eigenvalue correlations in the bulk of the spectrum and found [12] that they are given by the Dyson-Mehta-Wigner random matrix ensembles. If the two types of universal behavior are mutually inclusive, which we believe to be case, the microscopic spectral density of the Dirac operator is given by random matrix theory as well. Further arguments in favor of this conjecture can be found in [14]. General universality arguments [23] show that the properties of random matrix ensembles are not sensitive to specific shape of the distribution of the matrix elements. For convenience, and in agreement with the maximum entropy principle [16], we have chosen a gaussian distribution.

The microscopic spectral density of the corresponding chiral Gaussian Unitary Ensemble was obtained in [15, 10] and is given by

$$\rho_S(z) = \frac{z}{2}(J^2_{N_f+\nu}(z) - J_{N_f+\nu+1}(z)J_{N_f+\nu-1}(z)). \tag{7}$$

Here, N_f is the number of (massless) flavors and ν is the topological charge. Using (5) we are able to calculate the valence quark mass dependence of the chiral condensate. The integrals are known analytically [17] which leads to a remarkably simple result

$$\frac{\Sigma^V(u)}{\Sigma} = u(I_{N_f+\nu}(u)K_{N_f+\nu}(u) + I_{N_f+\nu+1}(u)K_{N_f+\nu-1}(u)), \tag{8}$$

where the I_ν and the K_ν are modified Bessel functions and

$$u \equiv mN\Sigma. \tag{9}$$

The valence quark mass dependence for zero, one and two massless flavors and topological charge $\nu = 0$ as given by eq. (8) is represented by the full lines in Fig. 2. Obviously, the lattice data are described by the curve with $N_f = \nu = 0$. In hindsight this is no surprise. First, in the region where the valence quark mass is much less than the sea quark mass, the fermion determinant has no bearing on the Dirac spectrum and we have effectively $N_f = 0$. Second, on a lattice the zero mode states and the much larger number of nonzero modes states are mixed, so that the effective topological charge that reproduces the lattice

results is equal to zero. The small u behavior of the valence quark mass dependence is $-u \log u$ for $N_f + \nu = 0$ and linear for $N_f + \nu \neq 0$. For valence quark masses smaller than the smallest eigenvalue of the Dirac operator over the *finite* ensemble of QCD gauge field configurations, a linear mass dependence is expected, which is indeed observed in lattice gauge calculations.

For comparison we have also shown the sea quark mass dependence of the chiral condensate (dashed line in Fig. 2). This can be obtained most conveniently from the finite volume partition function which follows from general symmetry arguments [18] or can be derived starting from the chiral random matrix ensemble [19]. The effective partition function describes the mass dependence of the QCD partition function for space time volumes

$$\frac{1}{\Lambda} \ll V^{1/4} \ll \frac{1}{m_\pi} \tag{10}$$

(Λ is a typical hadronic scale and $m_\pi \sim m\Lambda$ is the pion mass). It is straightforward to obtain the sea quark mass dependence of the chiral condensate from the effective partition function. In the sector of zero topological charge one finds [18]

$$\frac{\Sigma^S(u)}{\Sigma} = \frac{I_1(u)}{I_0(u)} \quad \text{for} \quad N_f = 1, \tag{11}$$

and

$$\Sigma^S(u) = \frac{I_1^2(u)}{u(I_0^2(u) - I_1^2(u))} \quad \text{for} \quad N_f = 2. \tag{12}$$

These results are shown by the dashed curves in Fig. 2 and should be compared to lattice data for the *sea* quark mass dependence of the chiral condensate which, unfortunately, are not available at this moment. For $m \ll \lambda_{\min}$ the mass in the determinant in not relevant and $\Sigma^c(u)$ should approach $\Sigma^V(u)$ which is indeed seen in Fig. 2.

For what range of masses can we expect the valence quark mass dependence to be described by random matrix theory? For $N_f \neq 0$, the small mass limit chiral random matrix theory reduces to the partition function of Leutwyler and Smilga [19], which gives the m-dependence of the QCD partition function in the range (10). The $N_f = 0$ limit can be obtained from results at arbitrary N_f by taking the limit $N_f \to 0$, in analogy with the replica trick in condensed matter physics[1]. Since the condition (10) is N_f independent

[1] Alternative, one may use a supersymmetric formulation to cancel the fermion determinant leading to supersymmetric quenched chiral perturbation theory [22].

it should be valid for $N_f = 0$ as well (with $m_\pi = m\Lambda$ and m equal to the valence quark mass). On the other hand, for lattice calculations to be physical, the pion should fit well inside the box,

$$\frac{1}{\sqrt{m_{\rm sea}\Lambda}} \ll L, \tag{13}$$

which leads to the condition

$$m \ll \frac{1}{L^2\Lambda} \ll m_{\rm sea}. \tag{14}$$

Because the pion is the lightest hadron the condition $\frac{1}{l}\Lambda \ll L$ in (10) is satisfied automatically. In practice the pion compton wave length is not *much* smaller than the length of the box. Indeed, the lattice data show a universal behavior up the valence quark masses close to the sea quark mass.

It is natural to expect that also the derivatives of the chiral condensate with respect to the mass (the scalar susceptibilities) show a universal scaling. Below we will discuss the following susceptibilities

$$\chi^{\rm dis} = \frac{1}{N}\partial_{m_f}\partial_{m_{f'}}\log Z = \langle\frac{1}{N}\sum_{k,l=1}^{N}\frac{1}{i\lambda_k+m}\frac{1}{i\lambda_l+m}\rangle - N\langle\frac{1}{N}\sum_{k=1}^{N}\frac{1}{\lambda_k^2+m^2}\rangle^2, \tag{15}$$

$$\chi^{\rm con} = \frac{1}{N}\partial_{m_f}\partial_{m_f}\log Z - \chi^{\rm dis} = -\langle\frac{1}{N}\sum_{k=1}^{N}\frac{1}{(i\lambda_k+m)^2}\rangle. \tag{16}$$

where the quark masses have been put equal to m after differentiation. The latter quantity is the isovector scalar susceptibility and $\chi^{\rm dis}$ enters in the isoscalar scalar susceptibility, $\chi^\sigma = 2\chi^{\rm dis} + \chi^{\rm con}$ (for two flavors).

As follows immediately from (16) $\chi^{\rm con}$ can be calculated from average spectral density (2). On the other hand, $\chi^{\rm dis}$ can be expressed into the connected two-point level correlation function

$$\rho_c(\lambda,\lambda') = \langle\rho(\lambda)\rho(\lambda')\rangle - \langle\rho(\lambda)\rangle\langle\rho(\lambda')\rangle. \tag{17}$$

If the eigenvalues are uncorrelated we have

$$\rho_c(\lambda,\lambda') = \langle\rho(\lambda)\rangle\delta(\lambda-\lambda'), \tag{18}$$

which leads to

$$\chi^{\rm dis} = \frac{\Sigma}{m}. \tag{19}$$

In random matrix theory, the eigenvalues are strongly correlated, which gives rise to a quite different prediction. We will consider the susceptibilities in the limit $mN\Sigma \gg 1$, but with $m \ll \Lambda$. If, at the same time, $m^2 N\Lambda^2 \ll 1$, the partition function of Leutwyler and Smilga or chiral random matrix theory can be used to calculate the susceptibilities [20]. In this limit the leading order contribution is expected to be of the form $1/m^2N$. In the physical limit, where the $m^2N\Lambda^2$ correction to the vacuum energy cannot be neglected, general symmetry arguments are insufficient to determine the susceptibility.

Since we are interested in the valence quark mass of the susceptibility we cannot use the effective partition function but have to rely on chiral random matrix theory. The valence quark mass dependence of $\chi^{\rm con}$ can be obtained from $\Sigma^V(u)$ given in eq. (8),

$$\chi^{\rm con} = \partial_m \Sigma^V(u) \sim \frac{7}{32}\frac{1}{m^2V}\frac{1}{u}, \tag{20}$$

where the asymptotic result is valid for $u \gg 1$ (the coefficient of the leading order term in $1u$ vanishes). For $N_f = 0$, $\chi^{\rm dis}$ can be obtained from the two level correlation function [15]. The result is given by

$$\chi^{\rm dis} \;=\; \frac{1}{4m^2V} F^{\rm dis}(u), \tag{21}$$

where both $F^{\rm dis}(u)$ can be expressed in modified Bessel functions and approaches 1 for $u \gg 1$.

In the limit $m^2N\Lambda^2 \gg 1$ the susceptibility is no longer determined by the lowest order terms in a chiral expansion of the vacuum energy. However, it is possible to calculate the susceptibilities in the framework of chiral random matrix theory but the results are model dependent. $\chi^{\rm con}$ can be obtained directly from the semicircular eigenvalue distribution. $\chi^{\rm dis}$ could only be obtained in an indirect way from the bosonized random matrix partition function. The results for all N_f are given by (for $m^2N\Lambda \gg 1$)

$$\chi^{\rm con} \;=\; -\frac{\Sigma^2}{2}, \tag{22}$$

$$\chi^{\rm dis} \;=\; 0. \tag{23}$$

The result for $\chi^{\rm dis}$ is a consequence of the stiffness of the random matrix spectra.

Unfortunately, no lattice data are available for the valence quark mass dependence of the susceptibilities quoted above. What has been calculated [21] is the dynamical quark

mass dependence of the susceptibilities. Because, in this way only relatively large quark masses have been studied, ultraviolet divergent terms make direct comparison difficult. In [21] $m^2V \approx 1$, but the numerical results show that, below the critical temperature, the quark mass dependence of the susceptibilities is $\sim 1/m$. One explanation is that the eigenvalues of the Dirac operator are only weakly correlated so that eq. (20) is applicable. On the other hand, spectral averaging shows strong correlations between the eigenvalues [12] implying that the ensemble and the spectral average of the eigenvalue correlations are not equal. This exciting possibility deserves further attention.

In conclusion, we have shown that the valence quark mass dependence of the chiral condensate is lattice QCD is given by a universal curve that can be described by chiral random matrix theory. As soon as lattice QCD results for the valence quark mass susceptibilities are available, we hope to extend our analysis to these observables as well.

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